JOINT QUANTUM COMMUNICATION AND SENSING

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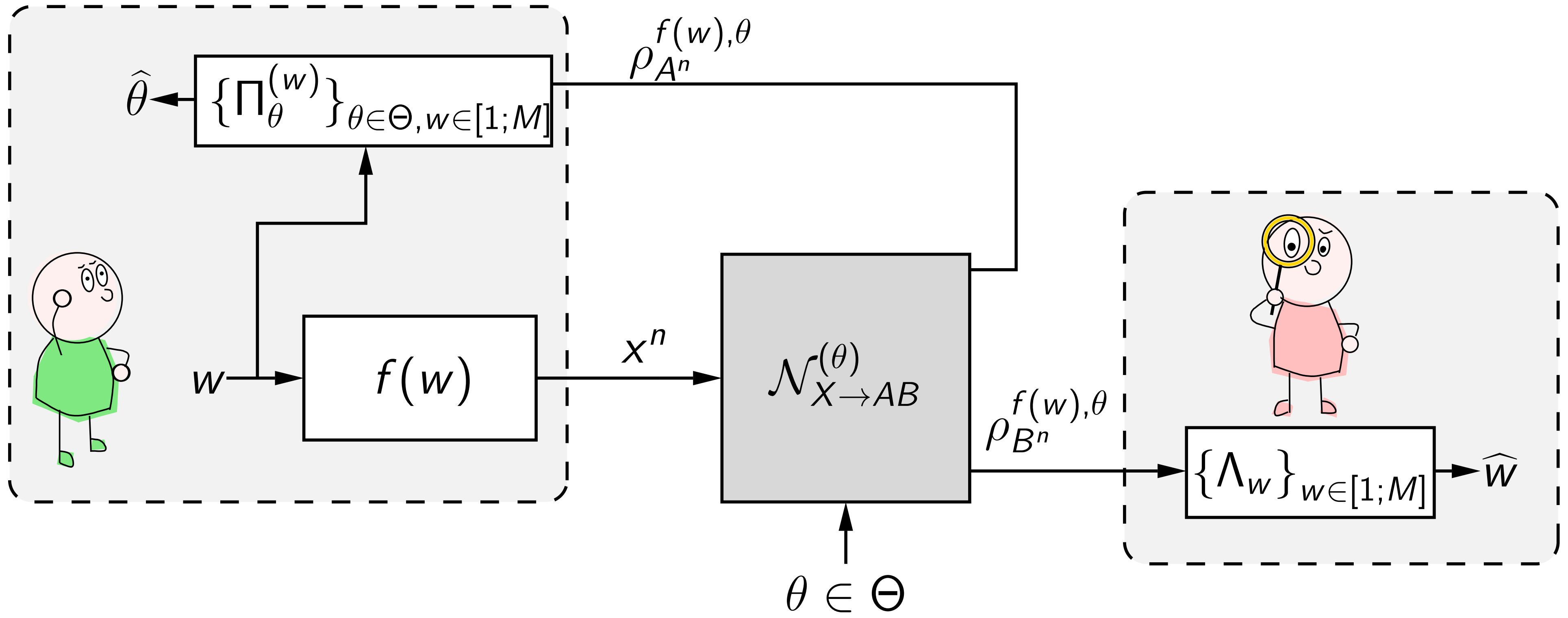
- **MOTIVATION**
  - mmWave systems enable convergence of radar and communication wavelengths
  - Previously separated communication and sensing systems can coexist on a single hardware
  - Sensing can significantly improve communication performance

- **USE CASES**
  - Object detection to assist beam steering
  - Geolocalization in GPS-denied environments
  - Autonomous driving, traffic monitoring, robotics and drone control

- **Present work:** investigate joint communication and sensing strategies in classical-quantum systems
  - What happens in a very lower power regime?

- **RELATED WORKS**
  - IID channel parameter and rate/distortion regime **[Zhang et al.'11, Kobayashi et al.'18'19, Ahmadipour et al.'21]**
  - Static channel parameter and rate/sensing-error exponent regime **[Chang et al. 22, Joudeh-Willems'22, Wu-Joudeh'22]**
  - Continuous channel parameter and rate/Cramer-Rao bound regime **[Xiong et al.'22]**
  - Quantum state discrimination **[Nussbaum-Szkoła'09'11, Li'16, Wilde et al.'20, Salek et al.'22]**



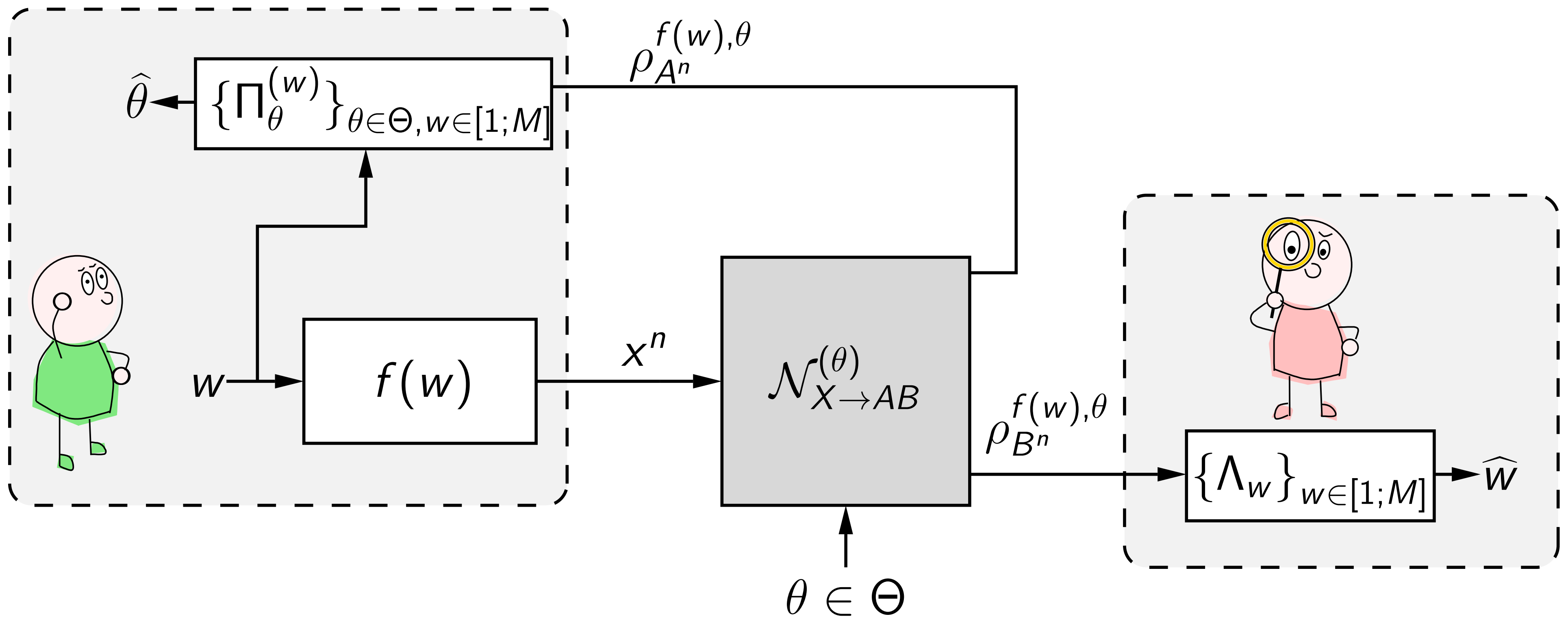
- **JOINT COMMUNICATION AND SENSING MODEL**
  - Parameter-dependent Classical-quantum channel (Compound channel)
  - Channel parameter $\theta \in \Theta, |\Theta| < \infty$ with prior probabilities $\{p_\theta\}_{\theta\in\Theta}$
  - Classical encoding function $f : [\![1, M]\!] \to \mathcal{X}^n$, Bob's decoding POVM $\{\Lambda_w\}_{w\in[\![1,M]\!]}$, Alice's collection of detection POVMs $\{\{\Pi_\theta^{(w)}\}_{\theta\in\Theta}\}_{w\in[\![1,M]\!]}$

- **PERFORMANCE METRIC**
  - Detection-error and communication-error probability

$$P_e^*\left(\{\{\Pi_\theta^{(w)}\}_{\theta\in\Theta}\}_{w\in[\![1,M]\!]}\right) \triangleq \max_{\theta\in\Theta} \max_{w\in[\![1,M]\!]} \operatorname{tr}\left((\mathbf{I} - \Pi_\theta^{(w)})\rho_{A^n}^{f(w),\theta}\right)$$

$$P_c^{(n)}\left(\{\Lambda_w\}_{w\in[\![1,M]\!]}\right) \triangleq \max_{\theta\in\Theta} \max_{w\in[\![1,M]\!]} \operatorname{tr}\left((\mathbf{I} - \Lambda_w)\,\rho_{B^n}^{f(w),\theta}\right)$$

  - Rate and detection-error exponent $\frac{1}{n}\log M$ and $E_{\mathrm{d}}^{(n)} \triangleq -\frac{1}{n}\log P_e^*$



THEOREM: RATE AND DETECTION-ERROR EXPONENT REGION

The region of rate/detection-error exponent is

$$\bigcup_{P_X\in\mathcal{P}_\mathcal{X}} \left\{ \begin{array}{l} (R,E)\in\mathbb{R}_+^2 : \\ R \leqslant \min_{\theta\in\Theta} \mathbb{I}\left(P_X, \mathcal{N}^{(\theta)}_{X\to B}\right) \\ E \leqslant \phi(P_X) \end{array} \right\}$$

where

$$\phi(P_X) = \min_{\theta} \min_{\theta'\neq\theta} \sup_{s\in[0,1]} \sum_x P_X(x)(1-s)\mathbb{D}_s(\rho_A^{x,\theta}\|\rho_A^{x,\theta'})$$

$$\mathbb{D}_s(\rho\|\sigma) \triangleq \frac{1}{s-1}\log\operatorname{tr}\left(\rho^s\sigma^{1-s}\right) \text{ and } \rho_{AB}^{x,\theta} = \mathcal{N}^{(\theta)}_{X\to AB}(x)$$

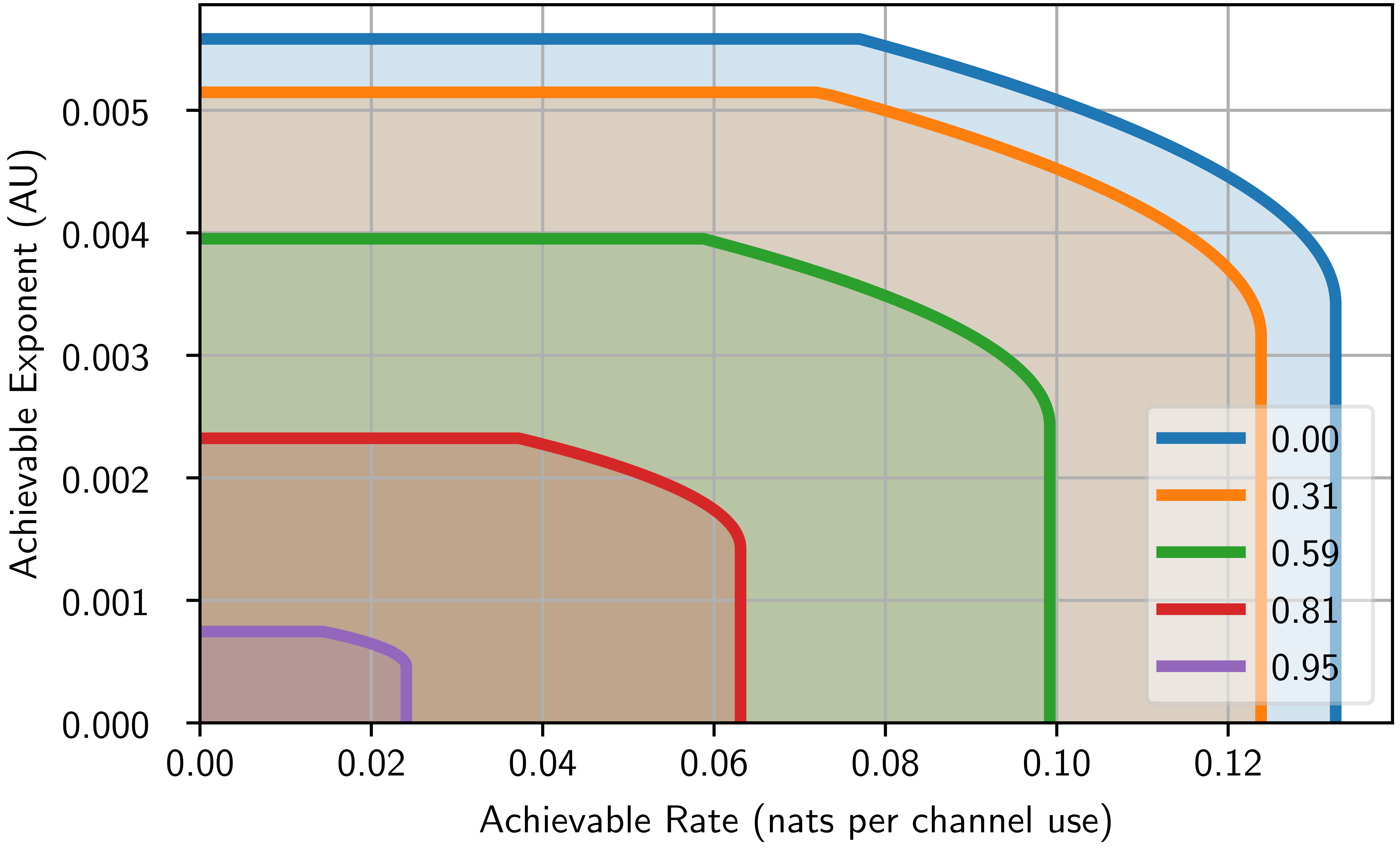
- Trade-off between rate and detection-error exponent is governed by (classical) type of codewords

- **Trade-off region**
  - Optimal error-exponent and optimal communication rate are achieved by different P_X
- **Numerical example**
  - $|\psi_0\rangle_A \triangleq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|\psi_1\rangle_A \triangleq R(\phi)\,|\psi_0\rangle$, where $R(\phi)$ is a rotation matrix.
  - Inner products of two states are indicated in legend

$$\rho_A^{x,\theta} = p_\theta^x(0)\,|\psi_0\rangle\langle\psi_0|_A + p_\theta^x(1)\,|\psi_1\rangle\langle\psi_1|_A$$

$p_\theta^x(0)$ for all x and θ.

θ \ x	0	1
0	0.9	0.3
1	0.9	0.2
2	0.7	0.2

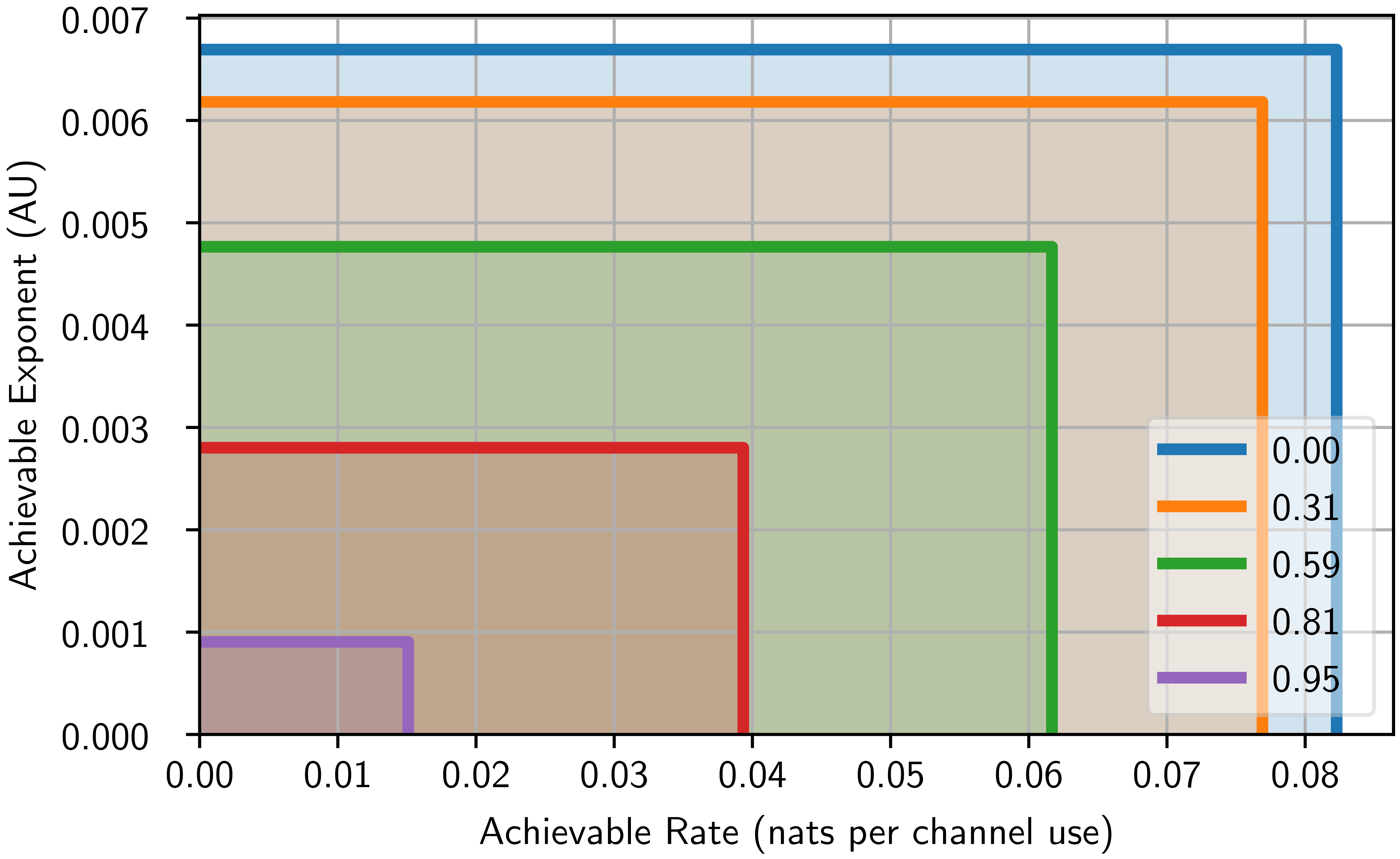


- **No trade-off scenario**
  - Optimal error-exponent and communication rate can be achieved by the same P_X

$$\rho_A^{x,\theta} = p_\theta^x(0)\,|\psi_0\rangle\langle\psi_0|_A + p_\theta^x(1)\,|\psi_1\rangle\langle\psi_1|_A$$

$p_\theta^x(0)$ for all x and θ.

θ \ x	0	1
0	0.9	0.1
1	0.8	0.2
2	0.7	0.3



DEFINITION: ACHIEVABILITY OF RATE AND EXPONENT PAIR

(R, E) is achievable if for every $\epsilon > 0$, there exists n large enough and a code $\mathcal{C}$ of length n , consisting of $(n, M, f, \{\{\Pi_\theta^{(w)}\}_{\theta\in\Theta}\}_{w\in[\![1,M]\!]}, \{\Lambda_w\}_{w\in[\![1,M]\!]})$, such that

$$P_{\mathrm{c}}^{(n)} \leqslant \epsilon, E_{\mathrm{d}}^{(n)} \geqslant E - \epsilon \quad \text{and} \quad \frac{1}{n}\log M \geqslant R - \epsilon.$$

- **SKETCH OF PROOF [Chang et al.'22]**
  - Fix type of codeword to P_X.
  - By **[Hayashi'09]**, there exists a sequence of constant-composition code with code rate $R < \min_{\theta\in\Theta} \mathbb{I}\left(P_X, \mathcal{N}_{X\to B}^{(\theta)}\right)$ such that $P_c^{(n)} \leqslant \epsilon$ for any positive ϵ and large enough n .
  - By specializing **[Li'16, Theorem 2]**, detection-error exponent of this code only depends on its type, i.e., $E_d^{(n)} \geqslant \phi(P_X) - \epsilon$ for any positive ϵ and large enough n .
  - Take union of all possible type P_X.

THEOREM: ONE-SHOT BAYESIAN-ERROR PROBABILITY BOUND [Li'16, Theorem 2]

Let $\sigma_1, \cdots, \sigma_r \in \mathcal{P}(\mathcal{H})$. For all $1 \leqslant i \leqslant r$, let $\sigma_i = \sum_{k=1}^{T_i} \lambda_{ik} Q_{ik}$ be the spectral decomposition of σ_i, and $T \triangleq \max\{T_1, \cdots, T_r\}$. There exists a function $f(r, T)$ and POVM $\{\Pi_i\}_{i \in [\![1,r]\!]}$ such that

$$\sum_{i=1}^{r} \operatorname{tr}\left(\sigma_i(\mathbf{I} - \Pi_i)\right) \leqslant f(r, T) \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \operatorname{tr}\left(Q_{ik} Q_{jl}\right),$$

where $f(r, T) < 10(r-1)^2 T^2$.

- **GOAL:** show that detection-error exponent only depends on type of codewords
- Specify result to $\sigma_\theta \triangleq p_\theta \bigotimes_{k=1}^{n} \rho_A^{x_k,\theta}$, where $\mathbf{x} = f(w)$, we have

$$\textstyle\sum_\theta \operatorname{tr}\left(p_\theta \rho_{A^n}^{\mathbf{x},\theta}(\mathbf{I} - \Pi_\theta^{(w)})\right) \leqslant f(|\Theta|, T) \sum_\theta \sum_{\theta' > \theta} \max(p_\theta, p_{\theta'}) \operatorname{tr}\left((\rho_{A^n}^{\mathbf{x},\theta})^s (\rho_{A^n}^{\mathbf{x},\theta'})^{1-s}\right)$$

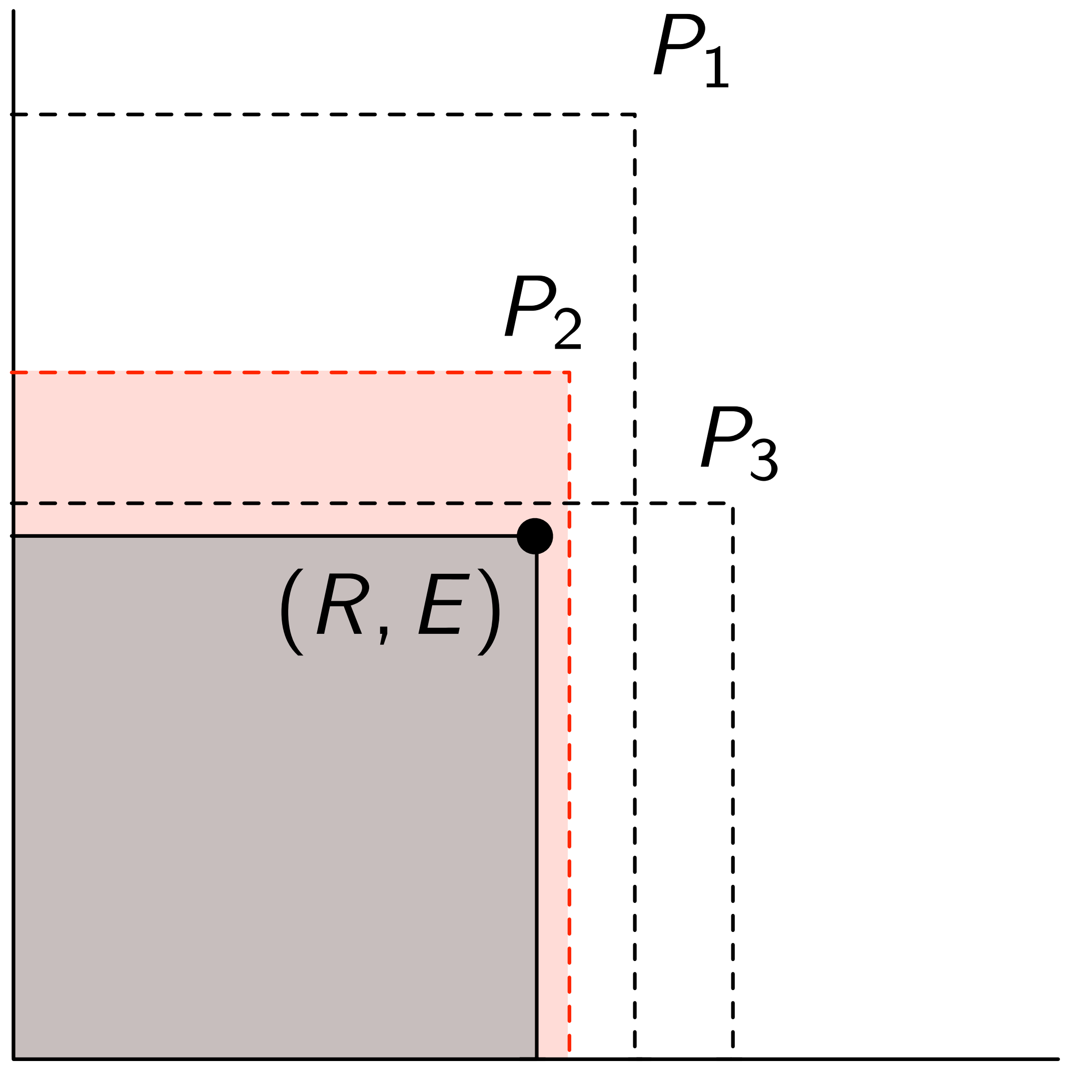
- Note that $\operatorname{tr}\left((\rho_{A^n}^{\mathbf{x},\theta})^s (\rho_{A^n}^{\mathbf{x},\theta'})^{1-s}\right) = \prod_{k=1}^{n} \operatorname{tr}\left((\rho_A^{x_k,\theta})^s (\rho_A^{x_k,\theta'})^{1-s}\right)$ by trace of tensor product state, we have

$$\log \operatorname{tr}\left((\rho_{A^n}^{\mathbf{x},\theta})^s (\rho_{A^n}^{\mathbf{x},\theta'})^{1-s}\right) = \textstyle\sum_u P_X(u) \log \operatorname{tr}\left((\rho_A^{u,\theta})^s (\rho_A^{u,\theta'})^{1-s}\right)$$

- With some manipulations, we can show that

$$\begin{aligned} -\frac{1}{n} \log P_e^*(\{\rho_{A^n}^{\mathbf{x},\theta}\}_\theta; \{\Pi_\theta^{(w)}\}_\theta) &\geqslant \min_\theta \min_{\theta' \neq \theta} \sup_{s \in [0,1]} -\sum_u P_X(u) \log \operatorname{tr}\left((\rho_A^{u,\theta})^s (\rho_A^{u,\theta'})^{1-s}\right) - \epsilon \\ &= \phi(P_X) - \epsilon \end{aligned}$$

- **KEY CHALLENGE**
  - Codebook is not of constant-composition, need to partition codebook into sub-codebooks according to their type.
  - Some of codeword types are more significant (i.e., have exponentially many codewords), and find upper bound on sizes of these **significant** sub-codebooks related to codeword type.
  - Derive upper-bound on the detection-error exponent based on codeword types.
  - Choose appropriate type representative of codebook and relate rate to detection-error exponent



- **SKETCH OF PROOF [Chang et al.'22]**
  - Rate analysis: any sub-codebooks of type P_X with exponentially many codewords must be upper-bounded by

$$\frac{\log M_{P_X}}{n} < \min_{\theta} \mathbb{I}(P_X, \mathcal{N}^{(\theta)}_{X\to B}) + \tau$$

  - Detection-error exponent analysis: for any codeword $\mathbf{x}$ with type P_X, detection-error exponent is upper-bounded by

$$-\frac{1}{n}\log P_e^*(\{\rho^{\mathbf{x},\theta}_{A^n}\}_{\theta\in\Theta}, \{\Pi_\theta\}_{\theta\in\Theta}) \leqslant \phi(P_X) + \delta$$

  - If (R, E) is achievable, given a codebook, for any type with exponentially many codewords

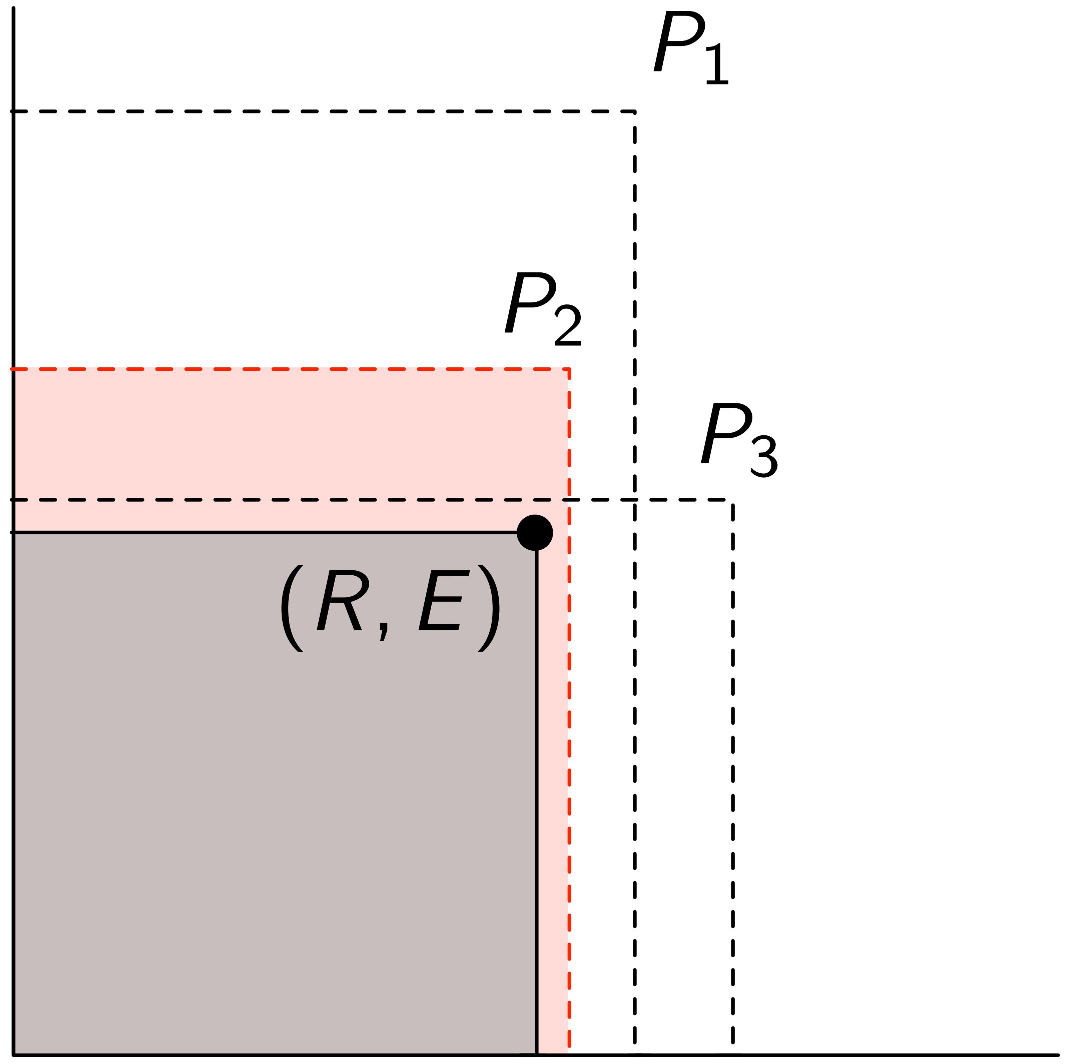
$$\frac{\log M_{P_X}}{n} > \frac{\log M}{n} - \delta \geqslant R - \epsilon - \delta \quad \text{and} \quad E - \epsilon \leqslant \phi(P_X) + \delta$$

we can choose $P_X^* \triangleq \operatorname{argmin}_{P_X\in\mathcal{T}} \phi(P_X)$, where $\mathcal{T}$ is set of type containing exponentially many codewords, such that

$$R \leqslant \min_{\theta} \mathbb{I}\left(P_X^*, \mathcal{N}^{(\theta)}_{X\to B}\right) + \tau + \epsilon + \delta$$

$$E \leqslant \phi(P_X^*) + \epsilon + \delta$$

  - Take union over all possible type



- Reduction to a binary quantum hypothesis testing **[Nussbaum-Szkoła'11]**
  - Fix a transmission codeword $\mathbf{x}$. Consider any POVM $\{\Pi_\theta\}_{\theta\in\Theta}$ for M-ary hypothesis testing, we construct a binary POVM for $(\rho^{\mathbf{x},\theta}_{A^n}, \rho^{\mathbf{x},\theta'}_{A^n})$ pair as follows: choose $A, B \succeq \mathbf{0}$, $A + B = \mathbf{I} - \Pi_\theta - \Pi_{\theta'}$, $\widetilde{\Pi}_\theta \triangleq \Pi_\theta + A$, $\widetilde{\Pi}_{\theta'} \triangleq \Pi_{\theta'} + B$.
  - By monotonicity of POVM, we have the lower bound on detection-error probability

$$\sum_{\theta\in\Theta} p_\theta \operatorname{tr}\left(\rho^{\mathbf{x},\theta}_{A^n}(\mathbf{I}-\Pi_\theta)\right) \geqslant \min\{p_\theta, p_{\theta'}\}\left(\operatorname{tr}\left(\rho^{\mathbf{x},\theta}_{A^n}(\mathbf{I}-\widetilde{\Pi}_\theta)\right) + \operatorname{tr}\left(\rho^{\mathbf{x},\theta'}_{A^n}\widetilde{\Pi}_\theta\right)\right)$$

- Reduction to a binary classical hypothesis testing **[Nussbaum-Szkoła'09]**
  - Optimal binary POVM to discriminate $(\rho^{\mathbf{x},\theta}_{A^n}, \rho^{\mathbf{x},\theta'}_{A^n})$ is Holevo-Helstrom test $\{\rho^{\mathbf{x},\theta'}_{A^n} - \rho^{\mathbf{x},\theta}_{A^n} > 0\}$, which is a Projection-Valued Measure (PVM).
  - WLOG, we analyze performance of any PVM and develop lower-bound accordingly.
  - Consider spectral decompositions:

$$\rho^{\mathbf{x},\theta}_{A^n} = \sum_{\mathbf{i}=(1,\cdots,1)}^{(|\mathcal{H}_A|,\cdots,|\mathcal{H}_A|)} \nu^{\mathbf{x}}_{\mathbf{i}} \left|\alpha^{\mathbf{x}}_{\mathbf{i}}\right\rangle\left\langle\alpha^{\mathbf{x}}_{\mathbf{i}}\right|, \text{ and } \rho^{\mathbf{x},\theta'}_{A^n} = \sum_{\mathbf{i}=(1,\cdots,1)}^{(|\mathcal{H}_A|,\cdots,|\mathcal{H}_A|)} \lambda^{\mathbf{x}}_{\mathbf{i}} \left|\beta^{\mathbf{x}}_{\mathbf{i}}\right\rangle\left\langle\beta^{\mathbf{x}}_{\mathbf{i}}\right|$$

  - Lower-bound obtained by binary classical hypothesis testing:

$$\operatorname{tr}\left((\mathbf{I}-\Gamma_\theta)\rho^{\mathbf{x},\theta}_{A^n}\right) + \operatorname{tr}\left(\Gamma_\theta\rho^{\mathbf{x},\theta'}_{A^n}\right) \geqslant \frac{1}{2}\left(\mathbb{P}_{P^{\mathbf{x}}_\theta}(P^{\mathbf{x}}_\theta(\mathbf{i},\mathbf{j}) < P^{\mathbf{x}}_{\theta'}(\mathbf{i},\mathbf{j})) + \mathbb{P}_{P^{\mathbf{x}}_{\theta'}}(P^{\mathbf{x}}_\theta(\mathbf{i},\mathbf{j}) \geqslant P^{\mathbf{x}}_{\theta'}(\mathbf{i},\mathbf{j}))\right)$$

where

$$P^{\mathbf{x}}_\theta(\mathbf{i},\mathbf{j}) = \prod_{k=1}^{n} \nu^{x_k}_{i_k}\left|\langle\alpha^{x_k}_{i_k}|\beta^{x_k}_{j_k}\rangle\right|^2, \text{ and } P^{\mathbf{x}}_{\theta'}(\mathbf{i},\mathbf{j}) = \prod_{k=1}^{n} \lambda^{x_k}_{j_k}\left|\langle\alpha^{x_k}_{i_k}|\beta^{x_k}_{j_k}\rangle\right|^2$$

- We derive the following lower-bound depending on type of codeword.

> **LEMMA:** LOWER-BOUND ON DETECTION-ERROR PROBABILITY [Nitinawarat et al.'13]
>
> Fix $(\rho_{A^n}^{\mathbf{x},\theta}, \rho_{A^n}^{\mathbf{x},\theta'})$. Then for any PVM $\{\Gamma_\theta, \mathbf{I} - \Gamma_\theta\}$ and $\xi > 0$, we have, for n large enough
>
> $$\mathrm{tr}\left((\mathbf{I} - \Gamma_\theta)\rho_{A^n}^{\mathbf{x},\theta}\right) + \mathrm{tr}\left(\Gamma_\theta \rho_{A^n}^{\mathbf{x},\theta'}\right) \geqslant \left(\frac{1}{2} - \xi\right) \exp\left(-n \sup_{s\in[0,1]} \sum_u P_X(u)(1-s)\mathbb{D}_s\left(\rho_A^{u,\theta} \middle\| \rho_A^{u,\theta'}\right)\right),$$
>
> where P_X corresponds to type of $\mathbf{x}$.

- Combining the above steps, we obtain

$$-\frac{1}{n}\log P_e^*(\{\rho_{A^n}^{\mathbf{x},\theta}\}_{\theta\in\Theta}, \{\Pi_\theta\}_{\theta\in\Theta}) \leqslant \phi(P_X) + \delta$$

- **CONCLUSION**
  - Studied the problem of joint communication and sensing over a c-q channel with unknown channel parameter.
  - Characterized an explicit rate/detection-error exponent region for non-adaptive strategy, similar to **[Chang et al.'22]**.
  - Tradeoff is governed by the empirical distribution of the codeword.

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